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A Modified Bioeconomic Model for Prey-Predator Interaction in Polluted Environment with Constant Harvesting: Reserve Zones and Taxation as Combine Control Strategy

Umar, A.* and Musa, S.
Department of Mathematics, Modibbo Adama University of Technology, P.M.B 2076, Yola – Nigeria
*Corresponding author: A. Umar, E-mail:ahmedumar42@gmail.com., Tel: 234-7033181737

Abstract-- A dynamical system of bioeconomic model for Prey-Predator interaction in a polluted surrounding with constant harvesting is proposed. The paper studied the dynamics of a fishery resource system in an aquatic environment that consists of two zones, a free fishing zone and a reserve zone where fishing is strictly prohibited. To protect fishery resource from extinction and over exploitation, taxation and creation of reserve zones are considered as combine control strategy in the proposed model. Interior equilibrium point is analysed alongside its local stability using linearization method and Routh-Hurwitz criteria for the proposed model. Optimal harvesting policy is formulated and solved with the help of Pontryagin's maximum principle. The numerical experiments illustrated confirm the theoretical results of the proposed model.

(Key words: bioeconomic, optimal harvesting, prey-predator, reserve zones, taxation, strategy, stability.)

1 INTRODUCTION

The increasing demand for more food and energy due to exceedingly growing of human population prompts exploitation of natural resources such as fish, forest and other renewable resources. Obviously, in an open-access fishery, the level of fishing effort expands or contracts accordingly as the economic rent to the fisherman is positive or negative. In that light, as fishing effort expands it translates to influx of more fishermen into the venture (fishing). This results to extensive and unregulated harvesting of fish resource, therefore, overexploitation of fish resources become evident. The eminent threat to fish resource caused by overexploitation has been a concern to fishery management to protect the ecosystem.

Some highlighted prominent regulatory policies that can sufficiently control the overexploitation of biological resources that fishery management has been yearning for was stressed by [10]. The regulatory policies includes: Taxation, License fees, Lease of property rights, seasonal harvesting etc. In the same vein, so many studies included creation of reserve zones among other regulatory policies. Out of such regulatory options, taxation is considered to be superior because of its economic flexibility contended by [10].

The continuous struggling to protect the fish resources, Prey-Predator and harvesting models play a crucial role in bioeconomics; that is the management of renewable resources as stressed by [1]. However, [6], [11], [12], [17], and some other authors have discussed the Prey-Predator system with harvesting. Similarly, other studies discussed the Prey-Predator system with harvesting and established marine reserve zones to protect the fish resource, such includes: [2]; [3]; [7]; [8]; [9]; [10]; [13]; [14]; [15]; [18];and [19].

Moreover, substantial studies have discussed Prey-Predator system with Harvesting by imposing taxation as the only control strategy which includes: [4]; [5]; [10] and [19]. Only few studies discussed Prey-Predator, harvesting and pollution model with reserve zones as the only control strategy as in [13] and [16].

From the literatures hitherto reviewed to the best of our knowledge, models considered by different authors considered only a control strategy. Moreso, authors who incooperated reserve zones in Prey-Predator with harvesting model assume Prey and Predator interaction in both reserve and unreserve zones. But in this paper, it is assumed that Prey and Predator interact only in unreserve zones. That assumption would save the biomass density of Prey from being decline by both harvesters and predators. Similarly in this paper, a model is proposed to combine reserve zones creation and taxation policy as strategies to regulate Prey-Predator interaction with constant harvesting and water pollution. The interior equilibrium point of system is established. The local stability of the interior equilibrium point of system using linearization and Routh-Hurwitz criteria for the proposed model is obtained. An optimal harvesting policy is also discussed using Pontryagin's maximum principle and numerical simulations illustrated.

2 MATERIALS AND METHODS

Consider a Prey-Predator system consisting of two zones: reserve and unreserve zones where fishing is only allowed in the unreserve zones. Each zone is assumed to be homogeneous, but interactions between Prey and Predator is allowed only in the unreserve zones. It is assumed that both Prey and Predator populations are harvested in unreserve zones. It is also assumed that migration of Prey and Predator is only from reserve to unreserve zones. In the absence of pollution, harvesting and Predation; the growth of both Prey and Predator is assumed to be logistic. Natural death rate of fish population in unreserve zone is relatively greater than that of reserve zone. For simplicity, it is assumed that the growth rate of prey is relatively greater than that of Predator in unreserve zone, while both Prey and Predator to have equal growth rate in reserve zone. Keeping these in view, the dynamic of the model is governed by the system of equations.

$$\left.\begin{aligned}
\dot{x}(t) &= \gamma_1\left(1-\frac{x}{k_1}\right)x - q_1E_1x - \frac{\beta_1 yx}{A+x} - d_1x - \sigma_1 x + \phi_1 x_R \\
\dot{y}(t) &= \gamma_2\left(1-\frac{y}{k_2}\right)y - q_2E_2y + \frac{\beta_2 yx}{A+x} - d_2y - \sigma_2 y + \phi_2 y_R \\
\dot{x}_R(t) &= \gamma_3\left(1-\frac{x_R}{k_3}\right)x_R - \phi_1 x_R - \sigma_3 x_R \\
\dot{y}_R(t) &= \gamma_4\left(1-\frac{y_R}{k_4}\right)y_R - \phi_2 y_R - \sigma_4 y_R \\
\dot{E}_T(t) &= \mu_1\left((p_1-\tau)q_1x - c_1\right)E_1 \\
\dot{E}_N(t) &= \mu_2\left((p_2-\tau)q_2y - c_2\right)E_2
\end{aligned}\right\}$$

(1)

$$x(0)=x^0,\ y(0)=y^0,\ E_T(0)=E_T^0,\ E_N(0)=E_N^0,\ x_R(0)=x_R^0,\ y_R(0)=y_R^0,\ \ t=0$$

(2) Let $x(t)$, $y(t)$, $x_R(t)$ and $y_R(t)$ represent biomass densities of the Prey and Predator population in the unreserve and reserve zones respectively at a time t. Let $E_T(t)$ and $E_N(t)$ represents the economic rent of the Prey and Predator population in the unreserve and reserve zones respectively at a time t. γ_1 and γ_3 are the intrinsic growth rate of Prey specie in the unreserve and reserve zones respectively. Similarly, γ_2 and γ_4 are the intrinsic growth rate of Predator specie in the unreserve and reserve zones respectively. k_1 and k_3 are the environmental carrying capacity for Prey specie in the unreserve and reserve zones respectively. Also, k_2 and k_4 are the environmental carrying capacity for Predator specie in the unreserve and reserve zones respectively. σ_1 and σ_3 are the natural death rate of Prey specie in the unreserve and reserve zones respectively. Equally, σ_2 and σ_4 are the natural death rate of Predator specie in the unreserve and reserve zones respectively. q_1 and q_2 are Catchability coefficient for Prey and Predator in the unreserve zones respectively. μ_1 and μ_2 are Stiffness parameter for Prey and Predator; d_1 and d_2 are death rate of Prey and Predator; p_1 and p_2 are constant price per unit biomass for Prey and Predator; c_1 and c_2 are constant cost per unit biomass for Prey and Predator; E_1 and E_2 are the

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harvesting effort for Prey and Predator; ϕ_1 and ϕ_2 are migration rate of Prey and Predator from reserve to unreserve zone respectively. β_1 is the maximal relative increase of Predation; β_2 is conversion factor from Prey to Predator; A is a saturation constant; and τ is a tax per unit biomass for Prey and Predator. All the parameters are assumed to be positive. To conserve the population of Prey-Predator system the regulatory policy imposes tax $\tau > 0$ per unit biomass of prey and predator, while $\tau < 0$ denotes the subsidies given to the fishermen.

2.1 EXISTENCE OF THE INTERIOR EQUILIBRIUM

The central focus of this study is to get interior equilibrium point of the model where all the biological species coexist. The interior equilibrium point of the model is found by equating the derivatives on the left hand sides of the system (1) to zero and solving the resulting algebraic equations simultaneously.

The interior equilibrium point of the model is obtained as follows:

$$P^* = \left(\omega_1, \omega_2, \frac{b_1 k_3}{\gamma_3}, \frac{b_2 k_4}{\gamma_4}, \frac{(A+\omega_1)\left[\gamma_3\left(a_1 k_1 + \gamma_1 \omega_1\right) + k_1 k_3 b_1 \phi_1\right] - k_1 \gamma_3 \beta_1 \omega_2}{k_1 q_1 \gamma_3 (A+\omega_1)}, \frac{(A+\omega_1)\left[\gamma_4\left(a_2 k_2 + \gamma_2 \omega_2\right) + k_2 k_4 b_2 \phi_2\right] + k_2 \gamma_4 \beta_2 \omega_1}{k_2 q_2 \gamma_4 (A+\omega_1)} \right) \quad \textbf{(3)}$$

Where $a_i = \left(\gamma_i - d_i - \sigma_i\right),\ i = 1,2.$

$b_i = \left(\gamma_j - \phi_i - \sigma_j\right),\ i = 1,2.\ \ j = 3,4.$

$\omega_i = \dfrac{c_i}{(p_i - \tau) q_i},\ i = 1,2.$

The equilibrium points P^* exist if and only if conditions in inequality (4) and (5) are satisfied

$$a_1,\ a_2,\ b_1,\ b_2 > 0 \quad \textbf{(4)}$$

$$\left.\begin{aligned} \beta_1 &< \frac{(A+\omega_1)}{k_1 \gamma_3 \omega_2}\left[\gamma_3\left(a_1 k_1 + \gamma_1 \omega_1\right) + k_1 k_3 \phi_1 b_1\right] \\ \beta_2 &> \frac{(A+\omega_1)}{k_2 \gamma_4 \omega_1}\left[\gamma_4\left(a_2 k_2 + \gamma_2 \omega_2\right) + k_2 k_4 \phi_2 b_2\right] \end{aligned}\right\}$$

(5)

2.2 LOCAL STABILITY ANALYSIS

Considering the local stability of the interior equilibrium point of the model. The variational matrix of the system (1) is obtained below:

$$J = \begin{pmatrix} a_1 - \frac{2\gamma_1}{k_1}x - q_1 E_1 + \frac{A\beta_1 y}{(A+x)^2} & -\frac{\beta_1 x}{(A+x)} & \phi_1 & 0 & -q_1 x & 0 \\ -\frac{A\beta_2 y}{(A+x)^2} & a_2 - \frac{2\gamma_2}{k_2}y - q_2 E_2 + \frac{\beta_2 x}{(A+x)} & 0 & \phi_2 & 0 & -q_2 y \\ 0 & 0 & b_1 - \frac{2x_R \gamma_3}{k_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 - \frac{2y_R \gamma_4}{k_4} & 0 & 0 \\ \mu_1 (p_1 - \tau) q_1 E_1 & 0 & 0 & 0 & \mu_1\left((p_1 - \tau) q_1 x - c_1\right) & 0 \\ 0 & \mu_2 (p_2 - \tau) q_2 E_2 & 0 & 0 & 0 & \mu_2\left((p_2 - \tau) q_2 y - c_2\right) \end{pmatrix} \quad \textbf{(6)}$$

Evaluating (6) at P^*, equation (7) is obtained as given below:

$$J^* = \begin{pmatrix} \frac{(A+\omega_1)^2\left[-3\gamma_1\gamma_3\omega_1 - k_1 k_3 b_1 \phi_1\right] - k_1\gamma_3\beta_1\omega_1\omega_2}{k_1\gamma_3 (A+\omega_1)^2} & -\frac{\beta_1\omega_1}{(A+\omega_1)} & \phi_1 & 0 & -q_1\omega_1 & 0 \\ -\frac{A\beta_2\omega_2}{(A+\omega_1)^2} & \frac{(A+\omega_1)\left[-3\gamma_2\gamma_4\omega_2 - k_2 k_4 b_2 \phi_2\right] + 2k_2\gamma_4\beta_2\omega_1}{k_2\gamma_4 (A+\omega_1)} & 0 & \phi_2 & 0 & -q_2\omega_2 \\ 0 & 0 & -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 0 \\ u_1 p_1\left[\frac{(A+\omega_1)(\gamma_3(a_1 k_1 + \gamma_1\omega_1) - k_1\gamma_3\beta_1\omega_2}{k_1\gamma_3 (A+\omega_1)}\right] & 0 & 0 & 0 & 0 & 0 \\ 0 & u_2 p_2\left[\frac{(A+\omega_1)(\gamma_4(a_2 k_2 + \gamma_2\omega_2) + k_2\gamma_4\beta_2\omega_1}{k_2\gamma_4 (A+\omega_1)}\right] & 0 & 0 & 0 & 0 \end{pmatrix} \quad \textbf{(7)}$$

The characteristic equation of the variational matrix J^* is given by

$$f(\lambda) = \lambda^6 + m_1\lambda^5 + m_2\lambda^4 + m_3\lambda^3 + m_4\lambda^2 + m_5\lambda + m_6$$

(8)

Where:

$$m_1 = (b_1 + b_2) - (s_1 + s_4)$$

$$m_2 = (b_1 + b_2)(1 - (s_1 + s_4)) - (s_1 s_4 + s_2 s_3) + q_1\omega_1 s_5 + q_2\omega_2 s_6$$

$$m_3 = (b_1 + b_2)\left[s_1 s_4 - s_2 s_3 - (s_1 + s_4) + q_1\omega_1 s_5 + q_2\omega_2 s_6\right] - (q_1\omega_1 s_4 s_5 + q_2\omega_2 s_1 s_6)$$

$$m_4 = (b_1 + b_2)(q_2\omega_2 s_1 s_6 - q_1\omega_1 s_4 s_5) + b_1 b_2 (s_1 s_4 - s_2 s_3 + q_1\omega_1 - q_2\omega_2 s_6) + q_1 q_2 \omega_1\omega_2 s_5 s_6$$

$$m_5 = (b_1 + b_2)(q_1 q_2 \omega_1\omega_2 s_5 s_6) + b_1 b_2 (q_2\omega_2 s_1 s_6 - q_1\omega_1 s_4 s_5)$$

$$m_6 = b_1 b_2 q_1 q_2 \omega_1\omega_2 s_5 s_6,$$

$$s_1 = \frac{(A+\omega_1)^2\left[-3\gamma_1\gamma_3\omega_1 - k_1 k_3 b_1 \phi_1\right] - k_1\gamma_3\beta_1\omega_1\omega_2}{k_1\gamma_3 (A+\omega_1)^2},$$

$$s_2 = -\frac{\beta_1\omega_1}{(A+\omega_1)}, \quad s_3 = -\frac{A\beta_2\omega_2}{(A+\omega_1)^2}$$

$$s_4 = \frac{(A+\omega_1)\left[-3\gamma_2\gamma_4\omega_2 - k_2 k_4 b_2 \phi_2\right] + 2k_2\gamma_4\beta_2\omega_1}{k_2\gamma_4 (A+\omega_1)},$$

$$s_5 = u_1 p_1\left[\frac{(A+\omega_1)(\gamma_3(a_1 k_1 + \gamma_1\omega_1) - k_1\gamma_3\beta_1\omega_2}{k_1\gamma_3 (A+\omega_1)}\right]$$

$$s_6 = u_2 p_2\left[\frac{(A+\omega_1)(\gamma_4(a_2 k_2 + \gamma_2\omega_2) + k_2\gamma_4\beta_2\omega_1}{k_2\gamma_4 (A+\omega_1)}\right]$$

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By the Routh-Hurwitz criterion, it follows that all eigenvalues of (8) have negative real parts if and only if (9) and (10) are satisfied.

$$(m_1m_4 - m_5)(m_1m_2m_3 - m_1^2 - m_1^2m_4) > m_5(m_1m_2 - m_3^2) + m_1m_1^5 \quad \textbf{(9)}$$

$$m_1^3m_3m_4(m_6 + m_1m_2m_5) + 2m_1m_5(m_4m_5 + m_1m_2m_6)m_5^2 + m_3(m_3^2m_6 + m_2m_5^2) > m_5^3 + m_1m_5(m_2^2m_5 + 3m_3m_6) + m_1m_6(m_2m_3^2 + m_1^2m_6) + m_4m_5(m_3^2 + m_1^2m_4) \quad \textbf{(10)}$$

Hence $P\left(x^*, y^*, x_R^*, y_R^*, E_T^*, E_N^*\right)$ is locally asymptotically stable. Therefore, the following theorem is obtained.

Theorem 3.1. *If equations (9) and (10) are satisfied, then the unique interior equilibrium point $P\left(x^*, y^*, x_R^*, y_R^*, E_T^*, E_N^*\right)$ of the system (1) is locally asymptotically stable.*

3 OPTIMAL HARVESTING POLICY

In renewable resources the fundamental problem is overexploitation, but the purpose of fishery management is planning harvests and keeping sustainable development of ecosystem. Such is achieved via determining the optimal trade-off between present and future harvests. An optimal harvesting policy to maximize the total discounted net revenue from the harvesting using taxation as a control instrument is designed.

In this section, the objective is to maximize the total discounted net revenues from the fishery given by

$$J = \int_0^\infty e^{-\delta t}\left(pqx - c\right)E\,dt$$

Where δ denotes the instantaneous annual rate of discount.

The objective is to determine a tax policy $\tau = \tau\left(t\right)$ to maximize J subject to the model equations in (1) and the control constraint

$$\tau_{\min} < \tau < \tau_{\max} \quad \textbf{(11)}$$

Applying Pontryangin's Maximum Principle to obtain the optimal equilibrium solution to this control problem. The Hamiltonian function is constructed by

$$\begin{aligned} H\left(x(t), y(t), x_R(t), y_R(t), E_1(t), E_2(t), \tau(t), t\right) &= e^{-\delta t}\left(pqx - c\right)E \\ &+ \lambda_1(t)\left[\gamma_1\left(1 - \frac{x}{k_1}\right)x - q_1E_1x - \frac{\beta_1 yx}{A + x} - d_1x - \sigma_1x + \phi_1x_R\right] \\ &+ \lambda_2(t)\left[\gamma_2\left(1 - \frac{y}{k_2}\right)y - q_2E_2y + \frac{\beta_2 yx}{A + x} - d_2y - \sigma_2y + \phi_2y_R\right] \\ &+ \lambda_3(t)\left[\gamma_3\left(1 - \frac{x_R}{k_3}\right)x_R - \phi_1x_R - \sigma_3x_R\right] + \lambda_4(t)\left[\gamma_4\left(1 - \frac{y_R}{k_4}\right)y_R - \phi_2y_R - \sigma_4y_R\right] \\ &+ \lambda_5(t)\left[\mu_1\left((p_1 - \tau)q_1x - c_1\right)E_1\right] + \lambda_6(t)\left[\mu_2\left((p_2 - \tau)q_2y - c_2\right)E_2\right] \end{aligned}$$

Where: $\lambda_i(t);\ i = 1, ..., 6$ are adjoint variables. Hamiltonian must be maximize for $\tau(t) \in \left[\tau_{\min}, \tau_{\max}\right]$. Specially, $\tau_{\min} < 0$ implies that subsidies have the effect of increasing the rate of expansion of the harvesting (Zhang, Zhang & Bai 2012). Assuming that the control constraints are not binding (i.e the optimal solution does not occur at $\tau(t) = \tau_{\min}$ or $\tau_{\max}$). The condition for a singular control to be optimal can be obtained by

$$\frac{\partial H}{\partial \tau} = 0, \text{ hence } \lambda_5 = \lambda_6 = 0 \quad \textbf{(12)}$$

For the adjoint equations

$$\frac{\partial \lambda_1(t)}{\partial t} = -\frac{\partial H}{\partial x} = -e^{-\delta t}pqE - \lambda_1(t)\left[\gamma_1 - \frac{2x}{k_1}\gamma_1 - q_1E_1 - \frac{A\beta_1 y}{(A + x)^2} - d_1 - \sigma_1\right] + \lambda_2(t)\left(\frac{\beta_2 x}{A + x}\right) \quad \textbf{(13)}$$

$$\frac{\partial \lambda_2(t)}{\partial t} = -\frac{\partial H}{\partial y} = -\lambda_1(t)\left(-\frac{\beta_1 x}{A + x}\right) - \lambda_2(t)\left[\gamma_2 - \frac{2y}{k_2}\gamma_2 - q_2E_2 + \frac{\beta_2 x}{(A + x)} - d_2 - \sigma_2\right] \quad \textbf{(14)}$$

$$\frac{\partial \lambda_3(t)}{\partial t} = -\frac{\partial H}{\partial x_R} = -\lambda_1(t)\phi_1 - \lambda_3(t)\left[\gamma_3 - \frac{2x_R}{k_3}\gamma_3 - \phi_1 - \sigma_3\right] \quad \textbf{(15)}$$

$$\frac{\partial \lambda_4(t)}{\partial t} = -\frac{\partial H}{\partial y_R} = -\lambda_2(t)\phi_2 - \lambda_4(t)\left[\gamma_4 - \frac{2y_R}{k_4}\gamma_4 - \phi_2 - \sigma_4\right] \quad \textbf{(16)}$$

$$\frac{\partial \lambda_5(t)}{\partial t} = -\frac{\partial H}{\partial E_1} = -e^{-\delta t}\left(pqx - c\right) - \lambda_1(t)q_1x \quad \textbf{(17)}$$

$$\frac{\partial \lambda_6(t)}{\partial t} = -\frac{\partial H}{\partial E_2} = -e^{-\delta t}\left(pqx - c\right) - \lambda_2(t)q_2y \quad \textbf{(18)}$$

From equations (17) and (18), we obtain

$$\lambda_1 = e^{-\delta t}\left(p - \frac{c}{qx}\right) \quad \textbf{(19)}$$

To obtain an optimal equilibrium solution, equation (19) substituted into (14) which can be rewritten as

$$\frac{\partial \lambda_2(t)}{\partial t} = A_2e^{-\delta t} - \lambda_2(t)A_1 \quad \textbf{(20)}$$

Where;

$A_1 = \left[\frac{2y}{k_2}\gamma_2 + q_2E_2 + d_2 + \sigma_2 - \frac{\beta_2 x}{(A+x)} - \gamma_2\right]$, $A_2 = \frac{\beta_1 x}{(A+x)}\left(p - \frac{c}{qx}\right)$

The solution of the linear equation in (20) is obtained below:

$$\lambda_2(t) = -\frac{A_2}{(A_1+\delta)}e^{-\delta t} \quad \textbf{(21)}$$

Similarly, taking (15) and (19), we have

$$\frac{\partial \lambda_3(t)}{\partial t} = A_4 e^{-\delta t} - \lambda_3(t)A_3 \quad \textbf{(22)}$$

Given that

$A_3 = \left(\frac{c}{qx} - p\right)\phi_1$ and $A_4 = \frac{2x_R}{k_3}\gamma_3 + \phi_1 + \sigma_3 - \gamma_3$

The solution of the linear equation in (22) is obtained below:

$$\lambda_3(t) = -\frac{A_3}{(A_4+\delta)}e^{-\delta t} \quad \textbf{(23)}$$

Substituting equations (21) into (13), we obtained

$$\frac{\partial \lambda_1(t)}{\partial t} = \lambda_1(t)B_1 - B_2 e^{-\delta t} \quad \textbf{(24)}$$

Where:

$B_1 = \frac{2x}{k_1}\gamma_1 + q_1E_1 + \frac{A\beta_1 y}{(A+x)^2} + d_1 + \sigma_1 - \gamma_1$ and

$B_2 = \frac{A_2\beta_2 x}{(A+\delta)(A+x)} + pqE$

The solution of the linear equation in (24) is obtained below:

$$\lambda_1(t) = \frac{B_2}{(B_1+\delta)}e^{-\delta t} \quad \textbf{(25)}$$

Substituting equations (19) into (25), we have

$$\left(p - \frac{c}{qx}\right) = \frac{B_2}{(B_1+\delta)} \quad \textbf{(26)}$$

Which provides an equation to the singular path and gives the optimal equilibrium levels of population $x^* = x_\delta,\ y^* = y_\delta,\ x_R^* = x_{(R)\delta},\ y_R^* = y_{(R)\delta},\ E_1^* = E_{1\delta},\ E_2^* = E_{2\delta}$. Then the optimal equilibrium levels of harvesting effort $\left(x_\delta, y_\delta, x_{(R)\delta}, y_{(R)\delta}, E_{1\delta}, E_{2\delta}\right)$ and the optimal tax is as $\tau_\delta = p - \frac{c}{qx_\delta}$.

4 NUMERICAL SIMULATIONS AND RESULTS

In this section Matlab R2010a is used to simulate numerical experiments with the help of parameter values given as

$\gamma_1 = 0.8$, $\gamma_2 = 0.65$, $\gamma_3 = 0.9$, $\gamma_4 = 0.9$, $k_1 = 600000$, $k_2 = 500000$, $k_3 = 600000$, $k_4 = 500000$, $\beta_1 = 0.000005$, $\beta_2 = 0.000003$, $A = 60000$, $d_1 = 0.2$, $d_2 = 0.2$, $\mu_1 = 0.1$, $\mu_2 = 0.12$, $p_1 = 750$, $p_2 = 700$, $c_1 = 500$, $c_2 = 500$, $\phi_1 = 0.5$, $\phi_2 = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.1$, $\sigma_4 = 0.1$, $E_1 = 1.20$, $E_2 = 1.50$, $q_1 = 0.000005$, $q_2 = 0.000012$, $\tau = 0.1$

For the above set of values of parameters, when tax is zero (tax not impose) we note that the positive equilibrium $P\left(x^*, y^*, x_R^*, y_R^*, E_T^*, E_N^*\right)$ exists and is given by

$x^* = 133,333$, $y^* = 59,524$, $x_R^* = 200,000$, $y_R^* = 166,666.67$, $E_T^* = 20,000,115,591$, $E_N^* = 66,754,772,177$

Again it was observed that when tax is imposed on the fishermen the positive equilibrium is given by $x^* = 200,000$, $y^* = 92,593$, $x_R^* = 200,000$, $y_R^* = 166,666.67$, $E_T^* = 20,000,133,903$, $E_N^* = 6,944,475,284.9$

Now plotting the dynamics of the system for the set of values of parameters with the help of Matlab R2010a. The behavior of x, y, x_R, y_R, E_T, and E_N with respect to time t is plotted in fig. 1. From this figure, we note that x, y, x_R and y_R increase for a very short time and then they decrease and finally settle down at its steady state. However, the economic rents E_T and E_N increase with time and attain their equilibrium level

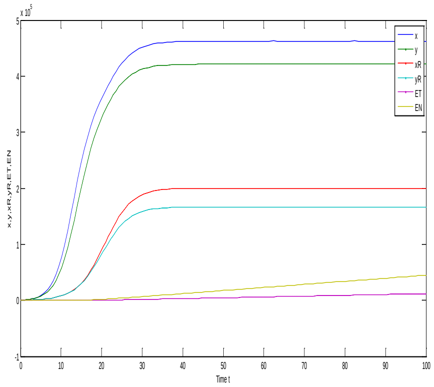

Fig. 1. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 0$, $\phi_1 = \phi_2 = 0.5$ holding other parameter values constant

Some other important parameters are ϕ_1 and ϕ_2 in the model. Varying these parameters tend to have a great impact on the biomass of the species in unreserved zones which are shown in fig. 2 and fig. 4. Increasing the migration rate from reserve zones to unreserve zones contemporaneously increases the biomass densities of species in unreserved zones as in fig. 1 through fig. 2 and fig. 3 through fig. 4. It was observed from the study that increase in the migration rate beyond $\phi_1 = \phi_2 = 0.7$ greatly affect the reserve zones. Hence, it is the optimal value for the migration rate as emphasized by [16].

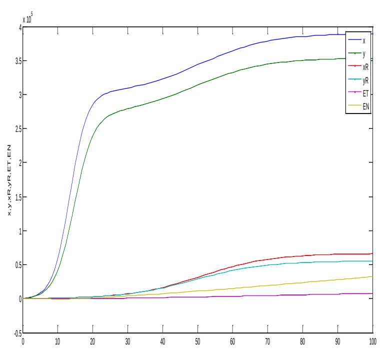

Fig. 2. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 0$, $\phi_1 = \phi_2 = 0.7$ holding other parameter values constant

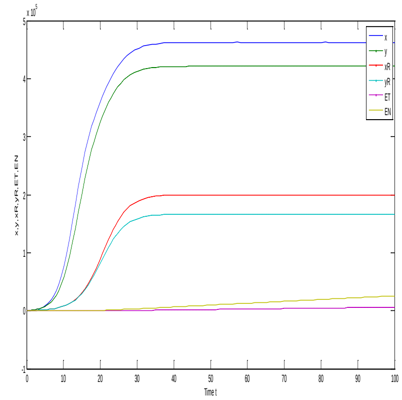

Fig. 3. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 250$, $\phi_1 = \phi_2 = 0.5$ holding other parameter values constant

Obviously, τ is also an important parameter which directs the dynamics of the system; However, in order to visualize the effect of imposing tax as a control measure in fishery, the behavior of x, y, x_R, y_R, E_T, and E_N with respect to time t for different values of τ are shown in fig. 3, fig. 5 through fig. 7. From these figures, it is evident that the densities of the resource biomass and population increase as τ increases, but the density of economic rent decreases as τ increases. For an optimal level of the tax imposed on per unit of harvested biomass, the resource biomass, the population and the economic rent settle down at their respective optimal level.

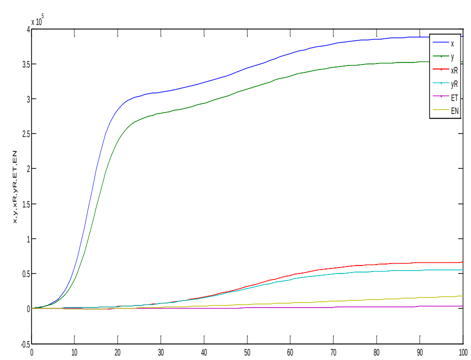

Fig. 4. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 250$, $\phi_1 = \phi_2 = 0.7$ holding other parameter values constant

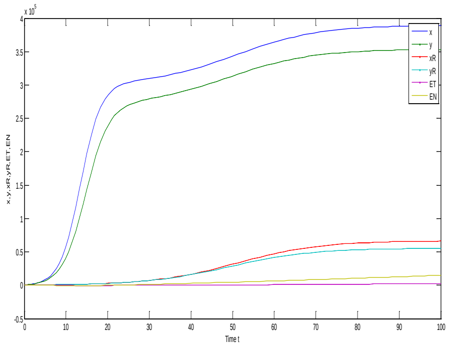

Fig. 5. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 300$, $\phi_1 = \phi_2 = 0.7$ holding other parameter values constant

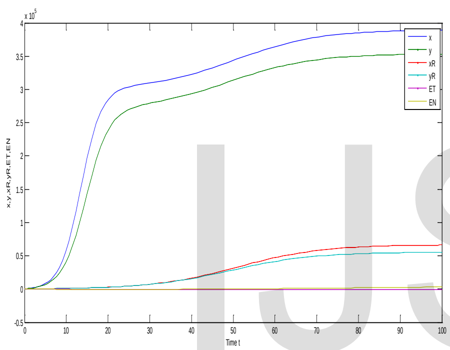

Fig. 6. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 500$, $\phi_1 = \phi_2 = 0.7$ holding other parameter values constant

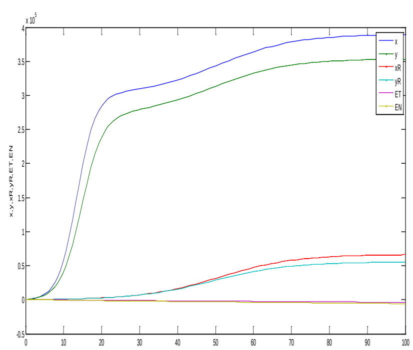

Fig. 7. Plot of x, y, x_R, y_R, E_T, and E_N versus time t when $\tau = 650$, $\phi_1 = \phi_2 = 0.7$ holding other parameter values constant

5 CONCLUSIONS

In this paper, a bioeconomic model of Prey-Predator interaction with constant harvesting in polluted environment has been discussed. The population densities of the biomass is partitioned into reserve and unreserve zones where fishing is only allowed in an unreserve zones. The study focuses its attention on the combined strategy using taxation and creation of reserve zones as an optimal governing mechanisms to control over exploitation of the fishery resource. Interior equilibrium point was analysed alongside the local stability using Routh-Hurwitz criteria for the proposed model. The study revealed that the proposed model is locally asymptotically stable. An optimal harvesting policy is also discussed using Pontryagin's maximum principle in the proposed model.

It has been observed from the numerical experiments that, increase in the migration rates from reserve zones to unreserve zones increase the densities of biomass in the unreserve zones which is evident in fig. 1 through fig. 2 and fig. 3 through fig. 4. Therefore, creating reserve zones protect the fishery resource from exploitation. As in the case of no taxation, even under continuous harvesting in the free fishing (unreserve) zone, the fish population may be maintained at an appropriate equilibrium level when reserve zones are created see fig. 1 and fig. 2. However, in the case of taxation, biomass densities of the fish species in both reserve and unreserve zones may be also sustained at an appropriate equilibrium level. But from fig. 4 through fig. 7, it is known that as the rate of tax increases both population densities of free fishing and reserve zones increase while economic rents decrease. This scenario depicts real life situation. Thus, combining taxation and creation of reserve zones is the optimal control strategy in curbing the phenomena of extinction and over exploitation of fishery resource.

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